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Numerical Solution of the Fokker-Planck Equations for a Hydrogen Plasma Formed by Neutral Injection[1]

JOHN KILLEEN AND ARCHER H. FUTCH

Lawrence Radiation Laboratory, University of California, Livermore, California 94551

ABSTRACT

We describe a program for the solution of the time-dependent Fokker–Planck equations for electron and ion distribution functions in velocity space. We consider the formation of a plasma in a magnetic mirror configuration by the injection of energetic neutral atoms. In the equations we have a source of energetic protons and cold electrons, and we include losses due to scattering and charge exchange. The coupled nonlinear partial differential equations are solved by an implicit difference method which is described. Numerical results are presented for two cases of interest.

INTRODUCTION

In those experiments in controlled fusion research that employ the injection of energetic neutral atoms into a magnetic mirror configuration, a plasma is formed of initially hot ions and cold electrons. It is of interest to know the velocity distribution functions of the electrons and ions as a function of time during the buildup of the plasma. The most suitable mathematical description is by means of the Fokker–Planck equations for the ion and electron distribution functions. This is because the dominant mechanism for energy transfer among the particles is by long range Coulomb interactions. The Fokker–Planck equations for the distribution functions of several species of particle, where the two-body force is an inverse-square law, have been derived in the paper of Rosenbluth, MacDonald, and Judd [1]. They use spherical polar coordinates in velocity space (v, θ, φ), where θ is the angle between the velocity vector and the magnetic field vector. They assume azimuthal symmetry so the resulting distribution functions are of the form $f(v, \theta, t)$. Calculations performed in this two-dimensional velocity space [2], [3] for the ion distribu-

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tion function indicate that good results can be obtained by separating the distribution function into a product of two terms. The first term is a function of v and t and the second term is a function of θ only. The equation for the function of θ is a Legendre differential equation on the domain $-\theta_c \leqslant \theta \leqslant \theta_c$, where θ_c defines the magnetic mirror loss cone. The equation for $f(v, t)$ must be solved numerically, and it is given in Eq. (6) of this paper for each species. The boundary condition on the distribution function in such a loss cone problem is $f(v, \theta_c, t) = 0$ for all v and t for each species which implies $f = 0$ at $v = 0$ in the separated solution. In those problems where we assume that the distribution functions are isotropic we take a symmetry condition at $v = 0$, i.e., $\partial f/\partial v = 0$ for all t.

Spatial dependence is not included in the model. In an earlier calculation [4], which included the finite orbit size of the ions and the spatial dependence of the trapping process it was found that the solutions for ion density exhibit growth rates similar to those obtained when the plasma density is assumed to be uniform.

In the equations for ions and electrons we include source terms which are appropriate for the neutral injection experiments such as ALICE [5] or PHOENIX [6]. We also include the loss of both species by scattering into the velocity space loss cone of the magnetic mirror configuration, and the hot ions can be lost by charge-exchange with the background gas.

A plasma potential is computed at each time step of the calculation by requiring charge neutrality. A critical velocity $v_{cr}(t)$ is determined such that electrons with $v < v_{cr}$ are not lost and those with $v > v_{cr}$ can be lost by scattering into the loss cone. At each time step the electron density is compared to the ion density and the velocity v_{cr} modified accordingly. The plasma potential is obtained from $e\varphi = \frac{1}{2}mv_{cr}^2$.

We have coupled nonlinear partial differential equations for the functions $f_e(v, t)$ and $f_i(v, t)$. We solve the equations numerically using finite difference methods. The equations are not linearized, i.e., the coefficients which involve moments of the distribution functions are computed at each time step. An implicit difference scheme is used, i.e., the velocity derivatives are replaced by difference quotients taken at the new time step while the coefficients are evaluated using the distribution function of the previous time step, and extrapolated. The scheme is stable numerically in practice with no restriction on the time step. This is an essential part of the calculations because as the electron temperature increases, the transfer rate decreases and the time step, Δt, must be continually increased during the calculation in order to progress toward equilibrium in a sensible manner.

In the last section we give numerical results for two cases of interest to the ALICE experiment [5], where we have used a source of 15-keV protons and 10-eV electrons and varied the beam current and background neutral density. In both of these cases an equilibrium density is reached; the electrons relax to a Maxwellian-type distribution.

The Fokker-Planck Equations for Ions and Electrons

In the paper of Rosenbluth, MacDonald, and Judd the Fokker-Planck equation for an inverse-square force is derived [1]. In Eq. (31) of that paper the result is given in spherical polar coordinates in velocity space, assuming azimuthal symmetry, i.e., no φ dependence. We further assume that the distribution functions are isotropic in velocity space, i.e., there is no θ dependence. The functions depend only on v, the magnitude of the velocity, and t, the time. Under this assumption the equation becomes

$$\Gamma_a^{-1}\frac{\partial f_a}{\partial t} = \frac{\partial^2 f_a}{\partial v^2}\left[\frac{1}{2}\frac{\partial^2 g}{\partial v^2}\right] + \frac{\partial f_a}{\partial v}\left[-\frac{\partial h_a}{\partial v} - \frac{1}{v^2}\frac{\partial g}{\partial v} + \frac{2}{v}\frac{\partial^2 g}{\partial v^2} + \frac{\partial^3 g}{\partial v^3}\right]$$
$$+ f_a\left[-\frac{2}{v}\frac{\partial h_a}{\partial v} - \frac{\partial^2 h_a}{\partial v^2} + \frac{2}{v}\frac{\partial^3 g}{\partial v^3} + \frac{1}{2}\frac{\partial^4 g}{\partial v^4}\right]. \tag{1}$$

The function $f_a(v, t)$ is the distribution function for particles of type a. The functions $h_a(v, t)$ and $g(v, t)$ are defined by the equations

$$h_a(v, t) = 4\pi \sum_b \frac{m_a + m_b}{m_b}\left[\int_0^v f_b(v', t)\frac{v'^2}{v}\,dv' + \int_v^\infty f_b(v', t)\,v'dv'\right], \tag{2}$$

$$g(v, t) = 4\pi \sum_b \left[\int_0^v f_b(v', t)\,v\left(1 + \frac{1}{3}\frac{v'^2}{v^2}\right)v'^2 dv'\right.$$
$$\left. + \int_v^\infty f_b(v', t)\left(1 + \frac{1}{3}\frac{v^2}{v'^2}\right)v'^3 dv'\right]. \tag{3}$$

The summations are taken over all the species of particles being considered, including type a. The constant Γ_a is defined by the equation

$$\Gamma_a \equiv (4\pi e^4/m_a^2)\ln D.$$

The quantity D is the ratio of the Debye length to the classical distance of closest approach.

The number density of particles of type a is given by

$$n_a(t) = 4\pi\int_0^\infty f_a(v, t)\,v^2 dv. \tag{4}$$

In Eq. (1) there are no source or loss terms, i.e., $dn_a/dt = 0$. We can consider the loss of particles by Coulomb scattering. The loss rate for such a process has been given by Chandrasekhar [7], and can be written as

$$\frac{dn_a}{dt} = -(4\pi)^2\int_0^\infty f_a(v, t)\,v^2\left[\sum_b \int_0^\infty k_a(v, v')\,f_b(v', t)\,v'^2 dv'\right]dv, \tag{5}$$

where

$$k_a(v, v') = p_a(v)\,\Gamma_a\left(1 + \frac{m_a}{m_b}\right)\frac{1}{v^3}\begin{cases}1 - \frac{1}{3}\frac{v'^2}{v^2}, & v \geqslant v',\\ \frac{2}{3}\frac{v}{v'}, & v \leqslant v',\end{cases}$$

and $p_a(v)$ is the probability that particles of type a and velocity v will be lost. We shall discuss the form of $p_a(v)$ that we use later. We can also add a source term, $s_a(v, t)$, to Eq. (1). The details of this term will be given in the next section.

If we use Eqs. (2) and (3) to evaluate the coefficients of Eq. (1), and also include the source and loss terms discussed above, then the equation for $f_a(v, t)$ becomes

$$\begin{aligned}(4\pi\Gamma_a)^{-1}\frac{\partial f_a}{\partial t} = {} & \frac{\partial^2 f_a}{\partial v^2}\left\{\sum_b\left[\frac{1}{3v^3}\int_0^v f_b(v', t)\,v'^4\,dv' + \frac{1}{3}\int_v^\infty f_b(v', t)\,v'\,dv'\right]\right\}\\ & + \frac{\partial f_a}{\partial v}\left\{\frac{1}{v}\sum_b\left[\frac{m_a}{m_b}\frac{1}{v}\int_0^v f_b(v', t)\,v'^2\,dv' - \frac{1}{3v^3}\int_0^v f_b(v', t)\,v'^4\,dv'\right.\right.\\ & \left.\left. + \frac{2}{3}\int_v^\infty f_b(v', t)\,v'\,dv'\right]\right\} + f_a\left(\sum_b \frac{m_a}{m_b} f_b\right)\\ & - f_a\left\{\frac{p_a(v)}{v^3}\sum_b\left[\left(1 + \frac{m_a}{m_b}\right)\int_0^v f_b(v', t)\,v'^2\,dv'\right.\right.\\ & \left.\left. - \frac{1}{3v^2}\int_0^v f_b(v', t)\,v'^4\,dv' + \frac{2v}{3}\int_v^\infty f_b(v', t)\,v'\,dv'\right]\right\} + s_a(v, t). \quad (6)\end{aligned}$$

The term for charge-exchange loss must be added to the above equation for ions. The term $s_a(v, t)$ represents the source of injected particles.

We consider electrons and ions of $Z = 1$. We introduce the dimensionless variable $x = v/v_0$, where v_0 is a constant and is a characteristic velocity. Let $f = (4\pi v_0^3/K_e) f_e$, where K_e is determined from the equation

$$n_e(0) = K_e\int_0^\infty f(x, 0)\,x^2\,dx;$$

i.e., the constant is determined by the initial conditions with $n_e(0)$ equal to the initial electron density. Similarly, we let $g = (4\pi v_0^3/K_i) f_i$, where

$$n_i(0) = K_i\int_0^\infty g(x, 0)\,x^2\,dx.$$

We introduce the dimensionless variable τ where $\tau = (\frac{1}{2}\Gamma_e K_e/v_0^3)\, t$. Let $\mu = m_e/m_i$ and $K = K_i/K_e$. We define functionals

$$M(f) = \int_x^\infty f(y, \tau)\, y\, dy, \tag{7}$$

$$N(f) = \int_0^x f(y, \tau)\, y^2\, dy, \tag{8}$$

and

$$E(f) = \int_0^x f(y, \tau)\, y^4\, dy. \tag{9}$$

In terms of these new variables, the equation for the electron distribution function becomes

$$\frac{\partial f}{\partial \tau} = A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial f}{\partial x} + Cf + D, \tag{10}$$

where

$$A = \frac{2}{3}\left\{\left[\frac{1}{x^3}E(f) + M(f)\right] + K\left[\frac{1}{x^3}E(g) + M(g)\right]\right\},$$

$$B = \frac{4}{3x}\left\{\left[\frac{3}{2x}N(f) - \frac{1}{2x^3}E(f) + M(f)\right] + K\left[\mu\frac{3}{2x}N(g) - \frac{1}{2x^3}E(g) + M(g)\right]\right\},$$

$$C = 2(f + K\mu g) - p_e(x)\frac{4}{3x^2}\left\{2\left[\frac{3}{2x}N(f) - \frac{1}{2x^3}E(f) + M(f)\right]\right.$$

$$\left. + K(1+\mu)\left[\frac{3}{2x}N(g) - \frac{1}{2x^3}E(g) + M(g)\right]\right\}.$$

The term $D(x, \tau)$ describes the time-dependent source of electrons.

The equation for the ion distribution function becomes

$$\frac{\partial g}{\partial \tau} = F\frac{\partial^2 g}{\partial x^2} + G\frac{\partial g}{\partial x} + Hg + L, \tag{11}$$

where

$$F = \frac{2}{3}\mu^2\left\{\left[\frac{1}{x^3}E(f) + M(f)\right] + K\left[\frac{1}{x^3}E(g) + M(g)\right]\right\},$$

$$G = \frac{4}{3x}\mu^2\left\{\left[\frac{1}{\mu}\frac{3}{2x}N(f) - \frac{1}{2x^3}E(f) + M(f)\right]\right.$$

$$\left. + K\left[\frac{3}{2x}N(g) - \frac{1}{2x^3}E(g) + M(g)\right]\right\},$$

and

$$H = 2\mu^2\left(\frac{1}{\mu}f + Kg\right) - H_1(x,\tau) - \mu^2 p_i(x)\,\frac{4}{3x^2}\left\{\frac{1}{2}\left(1+\frac{1}{\mu}\right)\left[\frac{3}{2x}N(f) - \frac{1}{2x^3}E(f) + M(f)\right] + K\left[\frac{3}{2x}N(g) - \frac{1}{2x^3}E(g) + M(g)\right]\right\}.$$

The term $H_1(x,\tau)$ contains the charge-exchange-loss term, and $L(x,\tau)$ describes the time-dependent source of ions.

At any time step we can determine the number density and average energy of each type of particle. Let $I_2^-(\tau)$ and $I_4^-(\tau)$ be the second and fourth moments of the electron distribution function, i.e.,

$$I_2^-(\tau) = \int_0^\infty f(x,\tau)\,x^2\,dx \tag{12}$$

$$I_4^-(\tau) = \int_0^\infty f(x,\tau)\,x^4\,dx. \tag{13}$$

The number density of electrons is given by

$$n_e(\tau) = K_e I_2^-(\tau) \tag{14}$$

and the mean electron energy is given by

$$E_e(\tau) = \frac{3}{2}kT_e = \frac{1}{2}m_e v_0^2\,\frac{I_4^-(\tau)}{I_2^-(\tau)}. \tag{15}$$

let

$$I_2^+(\tau) = \int_0^\infty g(x,\tau)\,x^2\,dx, \tag{16}$$

$$I_4^+(\tau) = \int_0^\infty g(x,\tau)\,x^4\,dx. \tag{17}$$

The number density of ions is given by

$$n_1(\tau) = K_1 I_2^+(\tau) \tag{18}$$

and the mean ion energy is given by

$$E_1(\tau) = \frac{3}{2}kT_1 = \frac{1}{2}m_1 v_0^2\,\frac{I_4^+(\tau)}{I_2^+(\tau)}. \tag{19}$$

Source and Loss Terms

In this section we shall describe the form of the source and loss terms that appear in Eqs. (10) and (11). The mechanisms for trapping the neutral-atom beam are the Lorentz ionization of excited hydrogen atoms and ionization of the neutral beam atoms by collisions with background gas molecules and with previously trapped ions and electrons. The growth rates of electrons and ions are given by the equations [4]

$$\frac{dn_e}{dt} = \frac{If^*}{V} + n_1\left[\frac{IL}{V}\left(\overline{\frac{\sigma_t^i v_r}{v_0}}\right) + \sigma_1^i v n_0\right] + n_e\left[\frac{IL}{V}\left(\overline{\frac{\sigma_t^e v_r}{v_0}}\right) + \sigma_1^e v n_0\right] \quad (20)$$

$$\frac{dn_1}{dt} = \frac{If^*}{V} + n_1\left[\frac{IL}{V}\left(\overline{\frac{\sigma_t^i v_r}{v_0}}\right)\right] + n_e\left[\frac{IL}{V}\left(\overline{\frac{\sigma_t^e v_r}{v_0}}\right)\right] \quad (21)$$

where I is the injected neutral beam current, V is the plasma volume, f^* is the fraction of the neutral beam ionized by the Lorentz force, L is the path length of the neutral beam through the plasma, v_r is the relative velocity between interacting particles, and v_0 is the characteristic velocity defined earlier which is determined by the beam velocity. The cross sections for ionization of the beam atoms by collisions with hot ions and electrons are σ_t^i and σ_t^e. Cross sections for ionization of the background gas by hot ions and electrons are σ_i^i and σ_i^e, and n_0 is the background gas density. In a magnetic mirror field, the cold ions (produced by charge-exchange collisions and by ionization of the background gas) may be neglected since they are rapidly scattered into the mirror escape cone and lost from the system, hence these terms are omitted from the above ion equation.

We assume that the injected electrons and ions have a velocity distribution defined by $S_e(x)$ and $S_i(x)$. Froms Eqs. (12) and (14) we have

$$\frac{dn_e}{d\tau} = K_e \int_0^\infty \frac{\partial f}{\partial \tau} x^2\, dx$$

and from Eqs. (16) and (18) we have

$$\frac{dn_1}{d\tau} = K_1 \int_0^\infty \frac{\partial g}{\partial \tau} x^2\, dx.$$

Using Eqs. (20) and (21) we can write

$$\frac{dn_e}{d\tau} = \left(\frac{t}{\tau}\right)\frac{1}{N(S_e)}\left(\frac{dn_e}{dt}\right)\int_0^\infty S_e(x)\, x^2\, dx$$

$$\frac{dn_1}{d\tau} = \left(\frac{t}{\tau}\right)\frac{1}{N(S_1)}\left(\frac{dn_1}{dt}\right)\int_0^\infty S_1(x)\, x^2\, dx,$$

where

$$N(S_e) = \int_0^\infty S_e(x)\, x^2\, dx$$

and

$$N(S_1) = \int_0^\infty S_1(x)\, x^2\, dx.$$

Hence in Eq. (10) we have

$$D(x, \tau) = S_e(x)\left(\frac{t}{\tau}\right) \frac{1}{K_e N(S_e)} \left(\frac{dn_e}{dt}\right) \tag{22}$$

and in Eq. (11) we have

$$L(x, \tau) = S_1(x)\left(\frac{t}{\tau}\right) \frac{1}{K_1 N(S_1)} \left(\frac{dn_1}{dt}\right) \tag{23}$$

In the calculations we keep $n_e(\tau) = n_i(\tau)$ by adjusting the plasma potential, so we can write Eqs. (22) and (23) as

$$D(x, \tau) = S_e(x)[l_0 + l_1 n_1(\tau)], \tag{24}$$

$$L(x, \tau) = S_1(x)[p_0 + p_1 n_1(\tau)], \tag{25}$$

where

$$l_0 = \left(\frac{t}{\tau}\right) \frac{1}{K_e N(S_e)} \frac{If^*}{V},$$

$$l_1 = \left(\frac{t}{\tau}\right) \frac{1}{K_e N(S_e)} \left[\frac{IL}{V}\left(\frac{\overline{\sigma_t^1 v_r}}{v_0} + \frac{\overline{\sigma_t^e v_r}}{v_0}\right) + n_0 v(\sigma_1^1 + \sigma_1^e)\right],$$

$$p_0 = \left(\frac{t}{\tau}\right) \frac{1}{K_1 N(S_1)} \frac{If^*}{V},$$

$$p_1 = \left(\frac{t}{\tau}\right) \frac{1}{K_1 N(S_1)} \left[\frac{IL}{V}\left(\frac{\overline{\sigma_t^1 v_r}}{v_0} + \frac{\overline{\sigma_t^e v_r}}{v_0}\right)\right],$$

and $n_i(\tau)$ is given by Eq. (18).

We can include up to ten sources of the type given by Eq. (25), corresponding to multiple ion beam injection at different energies. In the above discussion of source terms the (σv) terms were treated as constants; however, the cross sections have a velocity dependence determined by experimental measurements. We have polynomial descriptions for these functions so the terms $\sigma v n(\tau)$ can be replaced by integrals involving the distribution functions. This is illustrated in the next paragraph on charge exchange loss.

In Eq. (11) the charge-exchange loss term is $-H_1(x)\, g(x, \tau)$, where

$$H_1(x) = (t/\tau)\, n_0 v \sigma_{cx}(v) = (t/\tau)\, n_0 v_0 x \sigma_{cx}(x), \tag{26}$$

and $\sigma_{cx}(v)$ is the charge-exchange cross section. We have fit the experimental cross section, σ_{cx}, with a fifth-degree polynomial, so we write

$$H_1(x) = x[H_{a1} + H_{b1}x + H_{c1}x^2 + H_{d1}x^3 + H_{e1}x^4 + H_{h1}x^5] \tag{27}$$

where the coefficients are constants, including the constant factor, $(t/\tau)\, n_0 v_0$, which is an input parameter of the problem.

We shall now give the terms $p_e(x)$ and $p_i(x)$, which appear in Eqs. (10) and (11), and are the probabilities that electrons and ions of velocity $v_0 x$ are scattered into the loss cone of the magnetic mirror machine. In the case of no plasma potential these terms are equal and are given by Simon [8]. With a potential we follow the derivation by Kaufman [9] of the critical pitch angles in velocity space for the loss of ions and electrons. We denote the value of the magnetic field in the midplane by B_0, and the value at the mirror by $B_{\max}$. We consider a plasma potential which has value φ in the midplane and goes to zero at the mirror. Let W be the kinetic energy of a particle, then the total energy $H = W \pm e\varphi$ is a constant of the charged particle motion. We also assume that the magnetic moment $\lambda = W_\perp/B$ is a constant of the motion, where $W_\perp = \frac{1}{2}mv_\perp^2$ and $v_\perp$ is the component of velocity perpendicular to the magnetic field. The pitch angle in the midplane, α, is defined by

$$\sin^2 \alpha = \frac{W_\perp(\text{at } B_0)}{W(\text{at } B_0)} = \frac{\lambda B_0}{W(\text{at } B_0)} = \frac{B_0}{W(\text{at } B_0)} \frac{W_\perp(\text{at } B_{\max})}{B_{\max}}.$$

We define the critical pitch angle by the condition that $v_\parallel = 0$ at $B_{\max}$, and since $\varphi = 0$ at $B_{\max}$ we have

$$W_\perp(\text{at } B_{\max}) = H = W(\text{at } B_0) \pm e\varphi.$$

The critical pitch angle is then given by

$$\sin^2 \alpha_{\text{cr}}^{\pm} = \frac{B_0}{B_{\max}} \left(1 \pm \frac{e\varphi}{W(\text{at } B_0)}\right). \tag{28}$$

Electrons with energy $|W| < |e\varphi|$ are not lost and electrons with $|W| > |e\varphi|$ are lost with probability $p_e = 1 - \cos \alpha_{\text{cr}}$. Let the mirror ratio be given by $R = B_{\max}/B_0$, then we have

$$p_e(v) = \begin{cases} 1 - \sqrt{1 - \frac{1}{R}\left(1 - \frac{v_{\text{cr}}^2}{v^2}\right)}, & v \geqslant v_{\text{cr}} \\ 0, & v \leqslant v_{\text{cr}}. \end{cases} \tag{29}$$

The above expression is also used in the steady-state treatment of Fowler and Rankin [10].

The plasma potential is given by

$$e\varphi = \tfrac{1}{2}m_e v_{cr}^2 . \tag{30}$$

The procedure for determining v_{cr} is the following: At every time step $n_e(\tau)$ and $n_i(\tau)$ are computed from Eqs. (14) and (18), and the difference, $n_i(\tau) - n_e(\tau)$ is also computed. During the buildup of plasma, electrons tend to be lost faster than ions. Since we wish to keep $n_e(\tau) = n_i(\tau)$, the above difference is compared to a preassigned small number. If the difference exceeds this number, then v_{cr} is increased by an amount Δv_{cr} in order to decrease the electron loss rate and the time step is repeated. This process is repeated until $n_i(\tau) - n_e(\tau)$ is sufficiently small, and the calculation continues. As the plasma builds up and the electron energy increases the plasma potential also increases.

The term $p_e(x)$ is then

$$p_e(x) = \begin{cases} 1 - \left[1 - \dfrac{1}{R}\left(1 - \dfrac{x_{cr}^2}{x^2}\right)\right]^{1/2} & x \geqslant x_{cr} \\ 0 & x \leqslant x_{cr} \end{cases} \tag{31}$$

where $v_0 x_{cr} = v_{cr}$. The term $p_i(x)$ which appears in Eq. (11) is then

$$p_i(x) = 1 - \left[1 - \frac{1}{R}\left(1 + \frac{e\varphi}{\tfrac{1}{2}m_i v_0^2 x^2}\right)\right]^{1/2} \tag{32}$$

If the above square root becomes imaginary then $p_i(x) =$ a given constant.

The Difference Equations and Method of Solution

We wish to solve the two nonlinear differential equations (10) and (11) on the domain $0 \leqslant x \leqslant \infty$, $\tau \geqslant 0$, with the boundary conditions $f \to 0$, $g \to 0$ as $x \to \infty$, and $\partial f/\partial x = \partial g/\partial x = 0$ at $x = 0$ for $\tau > 0$, or in the separated solution case we have $f = g = 0$ at $x = 0$. The initial distributions $f(x, 0)$ and $g(x, 0)$ are given.

For the numerical solution we choose a domain $0 \leqslant x \leqslant x_J$, where x_J is specified for each problem and is taken large enough to include the high-velocity tail of the electron distribution. As the electrons increase in temperature, the distribution spreads out; thus the choice of x_J determines when the calculation must be stopped in order to preserve accuracy. At $x = x_J$, we take the boundary condition $f = g = 0$.

In the domain $0 \leqslant x \leqslant x_J$, $\tau \geqslant 0$, consider the finite-difference mesh defined by $x_j = j\Delta x$, $j = 0, 1, 2, \ldots, J$; $\tau^n = n\Delta\tau$, $n = 0, 1, 2, \ldots$. Let $f_j^n = f(x_j, \tau^n)$ and $g_j^n = g(x_j, \tau^n)$; $A_j^n = A(f_j^n, g_j^n, x_j, \tau^n)$, $B_j^n = B(f_j^n, g_j^n, x_j, \tau^n)$, etc. We define the first and second difference approximations by

$$(\delta f)_j^n = \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x},$$

$$(\delta^2 f)_j^n = \frac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{(\Delta x)^2}.$$

We approximate Eqs. (10) and (11) by the following implicit difference equations

$$\frac{f_j^{n+1} - f_j^n}{\Delta\tau} = \rho[A_j^{n+1}(\delta^2 f)_j^{n+1} + B_j^{n+1}(\delta f)_j^{n+1} + C_j^{n+1} f_j^{n+1} + D_j^{n+1}] + (1 - \rho)[A_j^n(\delta^2 f)_j^n + B_j^n(\delta f)_j^n + C_j^n f_j^n + D_j^n]$$

and

$$\frac{g_j^{n+1} - g_j^n}{\Delta\tau} = \rho[F_j^{n+1}(\delta^2 g)_j^{n+1} + G_j^{n+1}(\delta g)_j^{n+1} + H_j^{n+1} g_j^{n+1} + L_j^{n+1}] + (1 - \rho)[F_j^n(\delta^2 g)_j^n + G_j^n(\delta g)_j^n + H_j^n g_j^n + L_j^n]$$

where $\frac{1}{2} \leqslant \rho \leqslant 1$. We wish to solve these equations for the unknowns f_j^{n+1} and g_j^{n+1}; $j = 0, 1, 2, \ldots, J$.

We write the above difference equations as a set of simultaneous algebraic equations:

$$\alpha_j^{n+1} f_{j+1}^{n+1} - (1 + \beta_j^{n+1}) f_j^{n+1} + \gamma_j^{n+1} f_{j-1}^{n+1} = \psi_j^n, \tag{33}$$

$$\zeta_j^{n+1} g_{j+1}^{n+1} - (1 + \eta_j^{n+1}) g_j^{n+1} + \theta_j^{n+1} g_{j-1}^{n+1} = \varphi_j^n, \tag{34}$$

for $j = 1, 2, \ldots, J - 1$. The coefficients are defined by

$$\alpha_j^n = \rho \frac{\Delta\tau}{\Delta x}\left[\frac{A_j^n}{\Delta x} + \frac{1}{2} B_j^n\right],$$

$$\beta_j^n = \rho\Delta\tau\left[\frac{2A_j^n}{(\Delta x)^2} - C_j^n\right],$$

$$\gamma_j^n = \rho \frac{\Delta\tau}{\Delta x}\left[\frac{A_j^n}{\Delta x} - \frac{1}{2} B_j^n\right],$$

$$\psi_j^n = -\frac{1-\rho}{\rho}\alpha_j^n f_{j+1}^n - \left(1 - \frac{1-\rho}{\rho}\beta_j^n\right) f_j^n - \frac{1-\rho}{\rho}\gamma_j^n f_{j-1}^n$$
$$- \rho\Delta\tau D_j^{n+1} - (1-\rho)\Delta\tau D_j^n,$$

$$\zeta_j^n = \rho\frac{\Delta\tau}{\Delta x}\left[\frac{F_j^n}{\Delta x} + \frac{1}{2}G_j^n\right],$$

$$\eta_j^n = \rho\Delta\tau\left[\frac{2F_j^n}{(\Delta x)^2} - H_j^n\right],$$

$$\theta_j^n = \rho\frac{\Delta\tau}{\Delta x}\left[\frac{F_j^n}{\Delta x} - \frac{1}{2}G_j^n\right],$$

$$\varphi_j^n = -\frac{1-\rho}{\rho}\zeta_j^n g_{j+1}^n - \left(1 - \frac{1-\rho}{\rho}\eta_j^n\right) g_j^n - \frac{1-\rho}{\rho}\theta_j^n g_{j-1}^n$$
$$- \rho\Delta\tau L_j^{n+1} - (1-\rho)\Delta\tau L_j^n.$$

In the above coefficients we have

$$A_j^n = \frac{2}{3}\left\{\left[\frac{1}{x_j^3}E_j(f^n) + M_j(f^n)\right] + K\left[\frac{1}{x_j^3}E_j(g^n) + M_j(g^n)\right]\right\},$$

$$B_j^n = \frac{4}{3x_j}\left\{\left[\frac{3}{2x_j}N_j(f^n) - \frac{1}{2x_j^3}E_j(f^n) + M_j(f^n)\right]\right.$$
$$\left. + K\left[\mu\frac{3}{2x_j}N_j(g^n) - \frac{1}{2x_j^3}E_j(g^n) + M_j(g^n)\right]\right\},$$

$$C_j^n = 2\left\{f_j^n + K\mu g_j^n\right\} - (p_v)_j^n\frac{4}{3x_j^2}\left\{2\left[\frac{3}{2x_j}N_j(f^n) - \frac{1}{2x_j^3}E_j(f^n) + M_j(f^n)\right]\right.$$
$$\left. + K(1+\mu)\left[\frac{3}{2x_j}N_j(g^n) - \frac{1}{2x_j^3}E_j(g^n) + M_j(g^n)\right]\right\},$$

$$F_j^n = \mu^2 A_j^n,$$

$$G_j^n = \mu^2\frac{4}{3x_j}\left\{\left[\frac{1}{\mu}\frac{3}{2x_j}N_j(f^n) - \frac{1}{2x_j^3}E_j(f^n) + M_j(f^n)\right]\right.$$
$$\left. + K\left[\frac{3}{2x_j}N_j(g^n) - \frac{1}{2x_j^3}E_j(g^n) + M_j(g^n)\right]\right\},$$

$$H_j^n = 2\mu^2 \left\{\frac{1}{\mu} f_j^n + K g_j^n\right\} - \mu^2 (p_i)_j^n \frac{4}{3x_j^2} \left\{\frac{1}{2}\left(1 + \frac{1}{\mu}\right)\right.$$
$$\times \left[\frac{3}{2x_j} N_j(f^n) - \frac{1}{2x_j^3} E_j(f^n) + M_j(f^n)\right]$$
$$\left. + K\left[\frac{3}{2x_j} N_j(g^n) - \frac{1}{2x_j^3} E_j(g^n) + M_j(g^n)\right]\right\} - (H_1)_j^n .$$

In the above expressions the moments of the distribution functions are given by the following:

$$M_j(f^n) = \Delta x \left\{\frac{1}{2} f_j^n x_j + \sum_{i=j+1}^{J} f_i^n x_i\right\},$$

$$N_j(f^n) = \Delta x \left\{\frac{1}{2} f_j^n x_j^2 + \sum_{i=1}^{j-1} f_i^n x_i^2\right\},$$

$$E_j(f^n) = \Delta x \left\{\frac{1}{2} f_j^n x_j^4 + \sum_{i=1}^{j-1} f_i^n x_i^4\right\}.$$

In Eqs. (33) and (34) we have the unknowns f_j^{n+1}, g_j^{n+1}, $j = 1,..., J-1$ on the left-hand side of the equation, and the known quantities on the right-hand side. We are interested in solving these equations for the interior points, $j = 1,..., J-1$, since the boundary conditions at $x = 0$ and $x = x_J$ determine the solutions for $j = 0$ and $j = J$. Consequently, we do not have to worry about singularities at $x = 0$ in the coefficients. The system given by Eqs. (33) and (34) is non-linear in the unknowns f_j^{n+1}, g_j^{n+1}. If we extrapolate the coefficients, α_j^{n+1}, β_j^{n+1}, etc., from their values at the previous times τ^n, τ^{n-1}, then Eqs. (33) and (34) become a linear algebraic system in the unknowns f_j^{n+1}, g_j^{n+1}. The procedure is to extrapolate the coefficients and solve the linear system, then compute the coefficients α_j^{n+1}, β_j^{n+1}, etc., with the new values of f_j^{n+1}, g_j^{n+1}. This procedure works very well since the coefficients change in a very smooth manner with time.

We shall now give the method of solving the linearized equations. In Eq. (33) let

$$f_{j-1}^{n+1} = e_{j-1}^{n+1} f_j^{n+1} + d_{j-1}^{n+1}, \tag{35}$$

where e, d are to be determined. Then Eq. (33) becomes

$$\alpha_j^{n+1} f_{j+1}^{n+1} - (1 + \beta_j^{n+1}) f_j^{n+1} + \gamma_j^{n+1} e_{j-1}^{n+1} f_j^{n+1} + \gamma_j^{n+1} d_{j-1}^{n+1} = \psi_j^n$$

or

$$f_j^{n+1} = \frac{\alpha_j^{n+1} f_{j+1}^{n+1} + \gamma_j^{n+1} d_{j-1}^{n+1} - \psi_j^n}{1 + \beta_j^{n+1} - \gamma_j^{n+1} e_{j-1}^{n+1}} .$$

From Eq. (35) we can define

$$e_j^{n+1} = \frac{\alpha_j^{n+1}}{1 + \beta_j^{n+1} - \gamma_j^{n+1} e_{j-1}^{n+1}},$$

$$d_j^{n+1} = \frac{\gamma_j^{n+1} d_{j-1}^{n+1} - \psi_j{}^n}{1 + \beta_j^{n+1} - \gamma_j{}^{+1} e_{j-1}^{+1}}, \tag{36}$$

$$j = 1,..., J - 1.$$

In Eq. (34) let

$$g_{j-1}^{n+1} = a_{j-1}^{n+1} g_j^{n+1} + b_{j-1}^{n+1}; \tag{37}$$

then

$$a_j^{n+1} = \frac{\zeta_j^{n+1}}{1 + \eta_j^{n+1} - \theta_j^{n+1} a_{j-1}^{n+1}},$$

$$b_j^{n+1} = \frac{\theta_j^{n+1} b_{j-1}^{n+1} - \varphi_j{}^n}{1 + \eta_j^{n+1} - \theta_j^{n+1} a_{j-1}^{n+1}}, \tag{38}$$

$$j = 1, 2,..., J - 1.$$

From the boundary conditions at $x = 0$ we take $f_0^{n+1} = f_1^{n+1}$, $g_0^{n+1} = g_1^{n+1}$, so we have $e_0^{n+1} = 1$, $d_0^{n+1} = 0$, $a_0^{n+1} = 1$, $b_0^{n+1} = 0$ for all n. The computational procedure is to calculate e_j^{n+1}, d_j^{n+1}, a_j^{n+1}, b_j^{n+1}, $j = 1,..., J - 1$, from the recursion formulas (36) and (38); set $f_J^{n+1} = 0$, $g_J^{n+1} = 0$, and then calculate f_j^{n+1}, g_j^{n+1}, $j = 0, 1, 2,..., J - 1$ from Eqs. (35) and (37).

At each time step we compute the number density and average energy of the electrons and ions as given by Eqs. (14), (15), (18), and (19). The values of the integrals required are obtained from the functionals defined earlier, i.e., for $j = J$, we have

$$I_2^-(\tau^n) \equiv N_J(f^n), \qquad I_4^-(\tau^n) \equiv E_J(f^n),$$
$$I_2^+(\tau^n) \equiv N_J(g^n), \qquad I_4^+(\tau^n) \equiv E_J(g^n).$$

If Eqs. (10) and (11) are solved without source and loss terms, for an arbitrary initial distribution function the above integrals should all be constant and provide computational checks for the program. We find that the best results are obtained when the parameter ρ that appears in the difference equations is set equal to 1.

Numerical Results

A program of this type is used for a wide variety of parameter studies. In this section we show two cases of interest to illustrate the printed and plotted output.

We have investigated the formation of both hydrogen and deuterium plasmas. In the two cases to be shown we have a steady source of 15-keV protons and 10-eV electrons. In the first case the neutral beam current is 50 mA (3.1×10^{17} atoms/sec) and the background gas density is 5×10^7 molecules/cm^3, and in the second case the beam current 100 mA and the background gas density is 10^7 molecules/cm^3.

The velocity distribution of the ion source is $S_i(x) = e^{-100(x-1)^2}$, and for the electron source we take $S_e(x) = e^{-10(x-1)^2}$. We note that $x = 1$ corresponds to $v = v_0$, and we have $v_0 = 1.679 \times 10^8$ cm/sec. From Eqs. (15) and (19) the above distributions yield average energies of 15 keV for protons and 10 eV for electrons.

At $\tau = 0$ we have $n_e(0) = n_i(0) = 10^5$ particles/cm^3. We take $g(x, 0) = S_i(x)$ and for the initial electron distribution we use $f(x, 0) = 1.753e^{-1.2x^2}$ which also corresponds to 10-eV electrons. We have $K_e = 1.69 \times 10^5$ and $\Gamma_e = 1.5 \times 10^{19}$, hence $(t/\tau) = (2v_0^3/K_e\Gamma_e) = 3.7$.

In the source terms given by Eqs. (24) and (25) we can tabulate some of the parameters for the two cases.

	Case 1	*Case* 2
I	3.1×10^{17} atoms/sec	6.2×10^{17} atoms/sec
f^*	1.5×10^{-3}	1.5×10^{-3}
V	7.5×10^3 cm^3	7.5×10^3 cm^3
L	24.0 *cm*	24.0 cm
$\frac{If^*}{V}$	6.25×10^{10}	12.5×10^{10}
$\frac{IL}{V}$	10^{15}	2×10^{15}
n_0	5×10^7 molecules/cm^3	10^7 molecules/cm^3
v_0	1.679×10^8 cm/sec	1.679×10^8 cm/sec

In the terms for $p_e(x)$ and $p_i(x)$ given by Eqs. (31) and (32) we have used a mirror ratio, $R = 2.0$. The velocity domain for case 1 is $0 \leqslant x \leqslant 12.5$ with $\Delta x = 0.025$, and for case 2 we use $0 \leqslant x \leqslant 24.0$ with $\Delta x = 0.05$.

In Fig. 1 we show the ion density as a function of time for the two cases. In the first case the charge exchange loss is considerable and the buildup is to 4.2×10^9 ions/cm^3. The final density is determined by the scattering loss term; the first case was also run with a constant p_i, i.e.,

$$p_i = 1 - (1 - 1/R)^{1/2}$$

instead of Eq. (32), and the plasma density reached 10^{10} ions/cm^3. The inclusion

of the plasma potential in the expression for the ion loss cone has a substantial effect on the buildup. In the second case with a smaller background gas density the equilibrium plasma density is considerably enhanced.

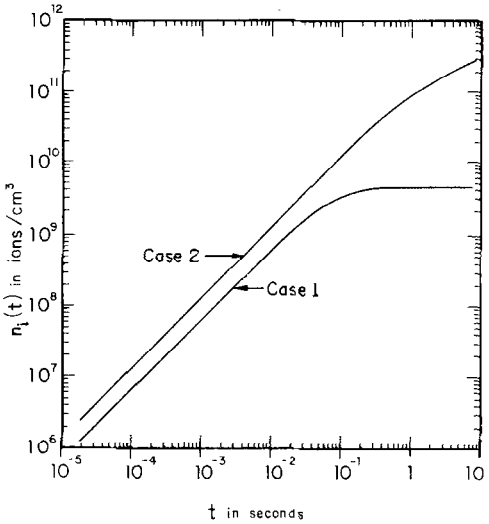

FIG. 1. Ion density as a function of time for cases 1 and 2.

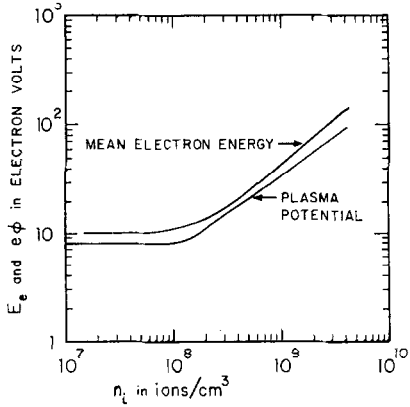

FIG. 2 (left). Mean electron energy and plasma potential as a function of ion density for case 1.

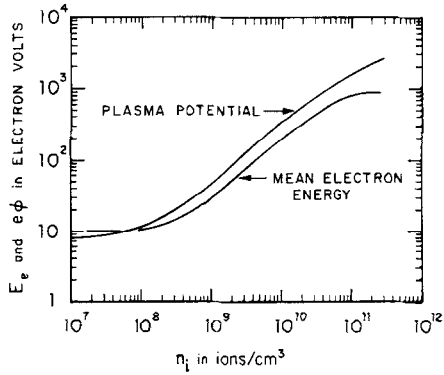

FIG. 3 (right). Mean electron energy and plasma potential as a function of ion density for case 2.

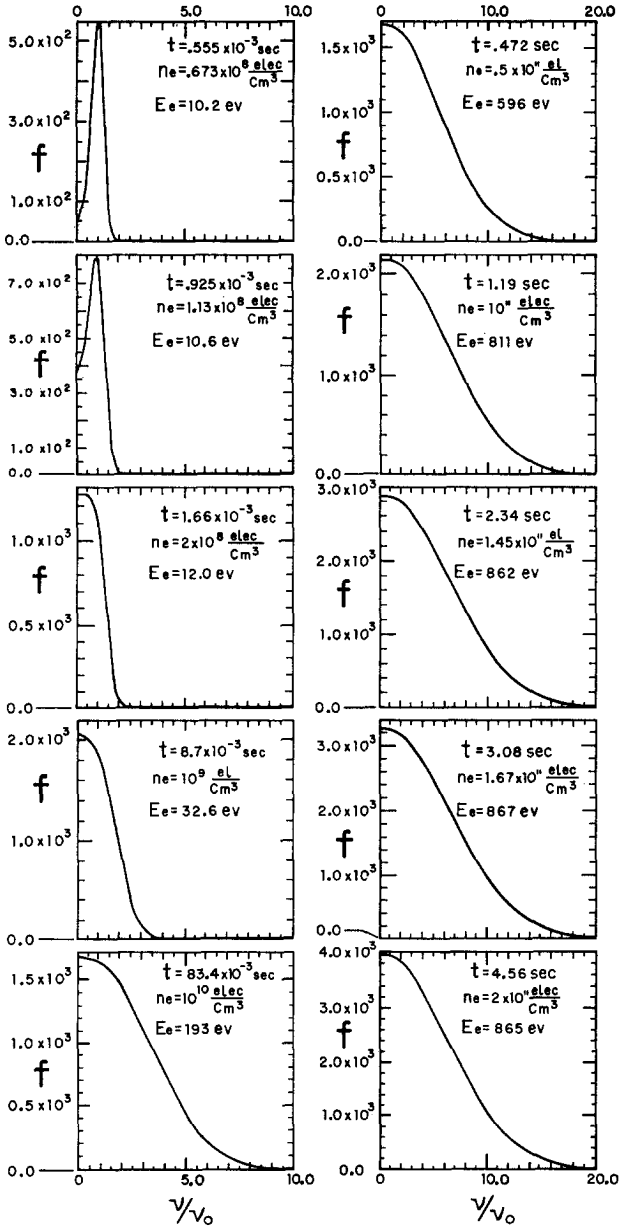

FIG. 4. Electron distribution functions as a function of x for various times for case 2.

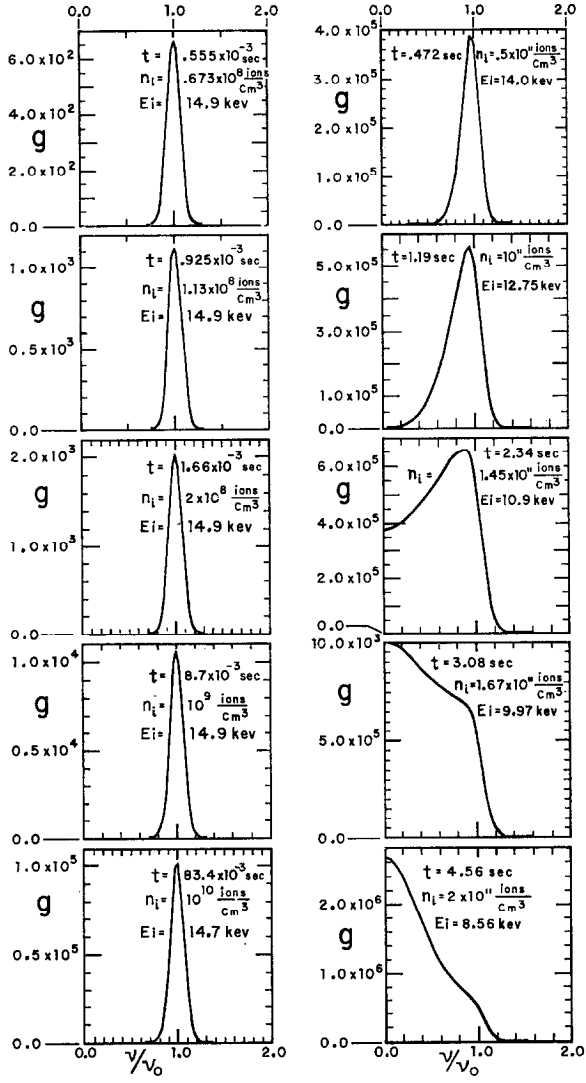

FIG. 5. Ion distribution functions as a function of x for various times for case 2.

In Figs. 2 and 3 we show the average electron energy and plasma potential in electron volts plotted as a function of ion density for the two cases. In the higher density case we see that the plasma potential required to achieve charge neutrality is considerably higher than the mean electron energy.

In Figs. 4 and 5 we show a sequence of plots of $f(x)$ and $g(x)$ at various times during the buildup of case 2. The plots for case 1 are similar, but in that case the ion distribution function $g(x, \tau)$ retains the shape of the source function with an amplitude proportional to the ion density. The electron distribution function starts out as a Maxwellian, but immediately take the shape of the source function, $S_e(x)$; it then later relaxes to a Maxwellian. The ion distribution function changes shape much more slowly, but we see that it assumes a similar distribution.

The curves of Figs. 4 and 5 are produced by the cathode-ray tube plotting routine of the program. In this case the ordinate scales of f and g are automatically changed. We can also fix the scales and make a movie showing the time development of the distribution functions.

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